Journal of Applied Mechanics and Technical Physics, Vol. 39, No. 1, 1998

OPTIMIZATION OF A LAYERED SPHERICAL INCLUSION IN A MATRIX IN TRIAXIAL TENSION AT INFINITY

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UDC 539.3

We consider the problem of synthesis, from a finite set of elastic homogeneous isotropic materials, of a multilayered spherical inclusion of minimum weight in a matrix stretched at infinity by three different uniform axial forces, under given restrictions on the strength and thickness of the inclusion. Necessary optimality conditions have been obtained, a computational algorithm has been constructed, and an example of calculation of an optimum inclusion has been given.

1. Formulation of the Problem. Let there exist a set W consisting of k homogeneous isotropic materials. This set is used to synthesize a layered spherical inclusion of minimum weight under given restrictions on the strength and thickness of the inclusion.

Let r_1 and r_2 be the radii of the inner and outer surfaces of an inclusion (see Fig. 1) located in a matrix stretched at infinity by three uniform axial forces q_1, q_2, and q_3. The pressure p is given at the internal boundary r_1. The stress-strain state (SSS) of the multilayered inclusion and the matrix in the spherical coordinate system $(r,\ \theta,\ \varphi)$ is described by a boundary-value problem that includes the equilibrium equations

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\sigma_{r\varphi}}{\partial\varphi}+\frac{1}{r}\left[2\sigma_{rr}-\sigma_{\theta\theta}-\sigma_{\varphi\varphi}+\sigma_{r\theta}\cot\theta\right]=0,$$
$$\frac{\partial\sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\sigma_{\theta\varphi}}{\partial\varphi}+\frac{1}{r}\left[3\sigma_{r\theta}+(\sigma_{\theta\theta}-\sigma_{\varphi\varphi})\cot\theta\right]=0, \tag{1.1}$$
$$\frac{\partial\sigma_{r\varphi}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\varphi}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\sigma_{\varphi\varphi}}{\partial\varphi}+\frac{1}{r}\left[3\sigma_{r\varphi}+2\sigma_{\theta\varphi}\cot\theta\right]=0,$$

the relations of Hooke's law

$$\sigma_{ij}=\frac{E}{1+\nu}\left[\frac{\nu}{1-2\nu}(\varepsilon_{kl}\delta_{kl})\,\delta_{ij}+\varepsilon_{ij}\right], \tag{1.2}$$

where the components of the strain tensor are expressed via radial $u_r(r,\ \theta,\ \varphi)$, meridional $u_\theta(r,\ \theta,\ \varphi)$, and circumferential $u_\varphi(r,\ \theta,\ \varphi)$ displacements in the form

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r},\quad \varepsilon_{\theta\theta}=\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r},\quad \varepsilon_{\varphi\varphi}=\frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}+\frac{u_r}{r}+\frac{u_\theta}{r}\cot\theta,$$
$$2\varepsilon_{r\theta}=r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial\theta},\qquad 2\varepsilon_{r\varphi}=\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\varphi}+r\frac{\partial}{\partial r}\left(\frac{u_\varphi}{r}\right), \tag{1.3}$$
$$2\varepsilon_{\theta\varphi}=\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{u_\varphi}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\varphi},$$

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0021-8944/98/3901-0127 $20.00

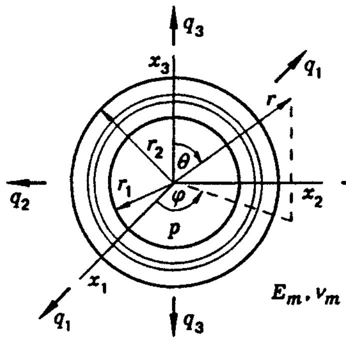

Fig. 1

and the boundary conditions

$$\begin{gathered}\sigma_{rr}(r_1,\,\theta,\,\varphi)=-p,\quad \sigma_{r\theta}(r_1,\,\theta,\,\varphi)=0,\quad \sigma_{r\varphi}(r_1,\,\theta,\,\varphi)=0,\\ \sigma_{rr}(\infty,\,\theta,\,\varphi)=q_1\sin^2\theta\cos^2\varphi+q_2\sin^2\theta\sin^2\varphi+q_3\cos^2\theta,\\ \sigma_{r\theta}(\infty,\theta,\varphi)=\sin\theta\cos\theta\,(q_1\cos^2\varphi+q_2\sin^2\varphi-q_3),\\ \sigma_{r\varphi}(\infty,\,\theta,\,\varphi)=(q_2-q_1)\sin\theta\sin\varphi\cos\varphi.\end{gathered}\tag{1.4}$$

Here $E(r)$ and $\nu(r)$ are Young's modulus and the Poisson ratio of the inclusion's layer materials and the matrix.

It is necessary to prescribe conjugation conditions (continuity of the displacements u_r, u_θ, and u_φ and the stresses σ_{rr}, $\sigma_{r\theta}$, and $\sigma_{r\varphi}$) at the internal boundaries $r_i \in (r_1,\,r_2]$ of the inclusion layers and at the inclusion-matrix boundary itself, where the characteristics of the medium undergo a discontinuity:

$$\begin{gathered}[u_r(r_i,\theta,\varphi)]=[u_\theta(r_i,\theta,\varphi)]=[u_\varphi(r_i,\theta,\varphi)]=0,\\ [\sigma_{rr}(r_i,\theta,\varphi)]=[\sigma_{r\theta}(r_i,\theta,\varphi)]=[\sigma_{r\varphi}(r_i,\theta,\varphi)]=0.\end{gathered}\tag{1.5}$$

Let σ, R, and ρ_* be quantities having the dimensionality of stress, length, and density, respectively. We introduce new dimensionless variables (later on, the asterisk is omitted):

$$u_i^*=\frac{u_i}{R},\quad r_i^*=\frac{r_i}{R},\quad \sigma_{ij}^*=\frac{\sigma_{ij}}{\sigma},\quad \sigma_s^*=\frac{\sigma_s}{\sigma},\quad E^*=\frac{E}{\sigma},\quad p^*=\frac{p}{\sigma},\quad q_i^*=\frac{q_i}{\sigma},\quad \rho^*=\frac{\rho}{\rho_*},$$

where σ_s and ρ are the strength and density limits of the materials from the set W. We make the substitution of coordinates

$$r=r_1+x(r_2-r_1),\quad x\in[0,\,1],\tag{1.6}$$

which transforms the variable domain $[r_1,\,r_2]$ into the constant domain $[0,\,1]$. We introduce the piecewise-constant function

$$\alpha(x)=\{\alpha_j;\,x\in[x_j,\,x_{j+1}\,),\,j=1,\ldots,n\},\quad x_1=0,\quad x_{n+1}=1,\tag{1.7}$$

which characterizes the structure of the layered inclusion, i.e., the number, dimensions, and materials of the constituent layers. The quantities α_j belong to the finite discrete set

$$U=\{\alpha_1,\ldots,\alpha_k\},\tag{1.8}$$

which corresponds to the given set of materials W. Now all the characteristics of the materials from the set W are the distribution functions $\alpha(x)$ on the closed interval $[0,\,1]$. It is convenient to give the set of integers $U=\{1,\ldots,k\}$ as U. After that, $\alpha(x)=i$, where $x\in[x_j,\,x_{j+1})$, means that the jth layer of the inclusion consists of the ith material from the set W.

Since the structure of the layered inclusion is determined by the function $\alpha(x)$ and the geometry is determined by the dimensions r_1 and r_2, we consider the pair $\{\alpha(x), r_1\}$ as a control (the outer radius r_2 can be assumed to be fixed without loss of generality), where $\alpha(x) \in U$ (1.8) and

$$0 < a \leqslant r_1 \leqslant b < r_2. \tag{1.9}$$

Here a and b are given limits within which the inner radius r_1 can vary.

The problem of optimum design consists in the following. Among the piecewise-constant functions $\alpha(x)$ (1.7) whose range of values belongs to the set U (1.8) and the parameters r_1 of the interval $[a, b]$, we need to find a control $\{\alpha(x), r_1\}$ that ensures a minimum of the weight functional

$$F[\alpha, r_1] = 4\pi \int\limits_{r_1}^{r_2} \rho(\alpha) r^2\, dr = \int\limits_0^1 G(\alpha, r_1, x)\, dx \tag{1.10}$$

for a given restriction on the tensile strength

$$\eta(x, \theta, \varphi, u_r, u_\theta, u_\varphi, \sigma_{rr}, \sigma_{r\theta}, \sigma_{r\varphi}, \alpha, r_1) \leqslant 0. \tag{1.11}$$

We regard the Mises yield condition as restriction (1.11):

$$\eta = (\sigma_{rr} - \sigma_{\theta\theta})^2 + (\sigma_{\theta\theta} - \sigma_{\varphi\varphi})^2 + (\sigma_{\varphi\varphi} - \sigma_{rr})^2 + 6\,(\sigma_{r\theta}^2 + \sigma_{r\varphi}^2 + \sigma_{\theta\varphi}^2) - 2\sigma_s^2 \leqslant 0. \tag{1.12}$$

Inequality (1.12) can be written in terms of u_r, u_θ, u_φ, σ_{rr}, $\sigma_{r\theta}$, and $\sigma_{r\varphi}$ using the Hooke's law relations (1.2).

2. Necessary Optimality Conditions. To derive necessary optimality conditions in problem (1.1)–(1.12), we need to obtain expressions for variations of the desired functional (1.10) and the restriction (1.12) by varying the control $\{\alpha(x), r_1\}$. With this in view, we transform the boundary-value problem (1.1)–(1.5). We introduce three spherical coordinate systems (r, θ_i, φ_i) $(i = 1, 2, 3)$, where the angle θ_i is counted off from the X_i axis of the Cartesian coordinate system (X_1, X_2, X_3). Figure 1 shows a coordinate system (r, θ, φ) that coincides in this case with the coördinate system (r, θ_3, φ_3). By virtue of the linear character of the elastic equations, the solution of the initial problem (1.1)–(1.5) can be represented as a superposition of four solutions. The first solution (Problem 1) describes the SSS of a layered spherical inclusion in an infinite matrix under the action of an internal pressure p. The remaining three solutions (Problem 2) determine the SSS of the inclusion in the matrix under the action of a uniform uniaxial force q_i along the X_i axis at infinity [1]. We consider the data of the problem.

Problem 1 includes the equilibrium equation

$$\frac{d\sigma_{rr}}{dr} + \frac{2}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0, \tag{2.1}$$

Hooke's law

$$\begin{aligned} \sigma_{rr} &= \frac{E}{1+\nu}\left[\frac{\nu}{1-2\nu}\left(\frac{du_r}{dr} + 2\frac{u_r}{r}\right) + \frac{du_r}{dr}\right], \\ \sigma_{\theta\theta} = \sigma_{\varphi\varphi} &= \frac{E}{1+\nu}\left[\frac{\nu}{1-2\nu}\left(\frac{du_r}{dr} + 2\frac{u_r}{r}\right) + \frac{u_r}{r}\right], \end{aligned} \tag{2.2}$$

and the boundary conditions

$$\sigma_{rr}(r_1) = -p, \qquad \sigma_{rr}(\infty) = 0. \tag{2.3}$$

The SSS of an inclusion-free matrix that is subject to condition (2.3) at infinity is described by the formulas

$$u_r = \frac{a}{r^2}, \quad \sigma_{rr} = -2\sigma_{\theta\theta} = -\frac{2E_m}{1+\nu_m}\frac{a}{r^3}, \tag{2.4}$$

where E_m and ν_m are Young's modulus and the Poisson ratio of the matrix material.

The conjugation conditions (1.5) at the internal boundaries of the inclusion layers and relations (1.6) make it possible to introduce the phase variables

$$\mathbf{Y}(x) = (u_r, \sigma_{rr})^t \tag{2.5}$$

on the interval [0, 1] (the superscript t means the transpose of the vector or matrix).

Using solution (2.4) for the matrix, problem (2.1)–(2.3) now can be represented as a boundary-value problem in the unknown $\mathbf{Y}(x)$ (2.5) only for the spherical inclusion:

$$\mathbf{Y}'(x) = A(\alpha, r_1, x)\mathbf{Y}(x), \quad y_2(0) = -p, \quad y_1(1) + \frac{r_2(1+\nu_m)}{2E_m} y_2(1) = 0. \tag{2.6}$$

Here the elements a_{ij} of the matrix $A(\alpha, r_1, x)$ are of the form

$$a_{11} = \frac{2\nu\,(r_2 - r_1)}{r(\nu - 1)}, \qquad a_{12} = \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}(r_2 - r_1),$$
$$a_{21} = \frac{2E(r_2 - r_1)}{r^2(1-\nu)}, \qquad a_{22} = \frac{(2-4\nu)(r_2 - r_1)}{r(\nu - 1)}.$$

We consider Problem 2. According to [2], we write the solution in the matrix and the spherical inclusion in the coordinate system (r, θ_i, φ_i) in the form

$$u_r(r, \theta_i) = q_i[u_1(r) + u_2(r)\cos 2\theta_i], \quad u_\theta(r, \theta_i) = q_i u_3(r)\sin 2\theta_i,$$
$$\sigma_{rr}(r, \theta_i) = q_i\,[\sigma_1(r) + \sigma_2(r)\cos 2\theta_i], \quad \sigma_{r\theta}(r, \theta_i) = q_i\,\sigma_3(r)\sin 2\theta_i \tag{2.7}$$

under the action of a uniform uniaxial force q_i along the X_i axis. Because Problem 2 is axisymmetric, all quantities do not depend on the coordinate φ_i, and the corresponding circumferential displacement u_φ, tangential stresses $\sigma_{r\varphi}$ and $\sigma_{\theta\varphi}$, and strains $\varepsilon_{r\varphi}$ and $\varepsilon_{\theta\varphi}$ are equal to zero. The nonvanishing components of the displacement vector and the stress and strain tensors are subject to conditions (1.1)–(1.3), and the boundary conditions (1.4) are reduced to the form

$$\sigma_{rr}(r_1, \theta_i) = \sigma_{r\theta}(r_1, \theta_i) = 0, \quad \sigma_{rr}(\infty, \theta_i) = \frac{q_i}{2}(1 + \cos 2\theta_i), \quad \sigma_{r\theta}(\infty, \theta_i) = -\frac{q_i}{2}\sin 2\theta_i. \tag{2.8}$$

The SSS of an inclusion-free matrix that is subject to condition (2.8) at infinity is described by the formulas [2]

$$u_1 = -\frac{a_1}{r^2} - \frac{3a_2}{r^4} + \frac{5-4\nu_m}{3(1-2\nu_m)}\frac{a_3}{r^2} + \frac{1-\nu_m}{2E_m}r, \quad u_2 = -\frac{9a_2}{r^4} + \frac{5-4\nu_m}{1-2\nu_m}\frac{a_3}{r^2} + \frac{1+\nu_m}{2E_m}r,$$
$$u_3 = -\frac{6a_2}{r^4} - \frac{2a_3}{r^2} - \frac{1+\nu_m}{2E_m}r, \quad \sigma_1 = \frac{2E_m}{1+\nu_m}\left[\frac{a_1}{r^3} + \frac{6a_2}{r^5} - \frac{5-\nu_m}{3(1-2\nu_m)}\frac{a_3}{r^3}\right] + \frac{1}{2}, \tag{2.9}$$
$$\sigma_2 = \frac{2E_m}{1+\nu_m}\left[\frac{18a_2}{r^5} - \frac{5-\nu_m}{1-2\nu_m}\frac{a_3}{r^3}\right] + \frac{1}{2}, \quad \sigma_3 = \frac{2E_m}{1+\nu_m}\left[\frac{12a_2}{r^5} - \frac{1+\nu_m}{1-2\nu_m}\frac{a_3}{r^3}\right] - \frac{1}{2}.$$

The conjugation conditions (1.5) and relations (1.6) and (2.7) allow us to introduce the continuous phase variables

$$\mathbf{Z}(x) = (u_1, u_2, u_3, \sigma_1, \sigma_2, \sigma_3)^{t} \tag{2.10}$$

on the interval [0, 1].

Using the solution (2.9) for the matrix, we now can represent Problem 2 [(1.1)–(1.3) and (2.8)] as a boundary-value problem in the unknown $\mathbf{Z}(x)$ (2.10) only for the spherical inclusion:

$$\mathbf{Z}'(x) = B(\alpha, r_1, x)\mathbf{Z}(x), \quad z_4(0) = z_5(0) = z_6(0) = 0,$$
$$\mathbf{Z}_f(1) = C(E_m, \nu_m)\mathbf{Z}_l(1) + \mathbf{D}(E_m, \nu_m). \tag{2.11}$$

Here $\mathbf{Z}_f(x) = (z_1, z_2, z_3)^t$ and $\mathbf{Z}_l(x) = (z_4, z_5, z_6)^t$; the nonzero elements b_{ij}, c_{ij}, and d_i of the matrices $B(\alpha, r_1, x)$ and $C(E_m, \nu_m)$ and the vector $\mathbf{D}(E_m, \nu_m)$ are of the form

$$b_{11} = 2b_{13} = b_{22} = \frac{2}{3}b_{23} = -b_{65} = \frac{2\nu(r_2 - r_1)}{r(\nu - 1)}, \quad b_{14} = b_{25} = \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}(r_2 - r_1),$$

$$b_{36} = \frac{2(1+\nu)}{E}(r_2 - r_1), \quad \frac{1}{2} b_{32} = b_{33} = -b_{46} = -\frac{1}{3} b_{56} = -\frac{1}{3} b_{66} = \frac{r_2 - r_1}{r},$$

$$b_{41} = 2b_{43} = b_{52} = \frac{2}{3} b_{53} = b_{62} = \frac{2E(r_2 - r_1)}{r^2(1-\nu)}, \quad b_{65} = \frac{E(5+\nu)(r_2 - r_1)}{r^2(1-\nu^2)},$$

$$b_{44} = b_{55} = \frac{(2-4\nu)(r_2 - r_1)}{r(\nu - 1)}, \quad a = \frac{r_2(1+\nu_m)}{4E_m(7-5\nu_m)},$$

$$c_{11} = -\frac{r_2(1+\nu_m)}{2E_m}, \quad c_{12} = a\,(3\nu_m - 1), \quad c_{13} = a\,(5 - 7\nu_m),$$

$$c_{22} = a\,(19\nu_m - 17), \quad c_{23} = a\,(15 - 21\nu_m), \quad c_{32} = a\,(10 - 14\nu_m),$$

$$c_{33} = a\,(26\nu_m - 22), \quad d_1 = 24a\,\frac{1-\nu_m}{1+\nu_m}, \quad d_2 = -d_3 = 30a\,(1-\nu_m).$$

The stress-tensor components in the restriction on strength (1.12) are expressed, in the original spherical coordinate system (r, θ, φ), via the solutions $\mathbf{Y}(x)$ and $\mathbf{Z}(x)$ of the boundary-value problems (2.6) and (2.11) as

$$\sigma_{rr}(r,\theta,\varphi) = y_2 + (z_4 - z_5)(q_1 + q_2 + q_3) + 2z_5[q_3 \cos^2\theta + (q_1 \cos^2\varphi + q_2 \sin^2\varphi)\sin^2\theta],$$

$$\sigma_{\theta\theta}(r,\theta,\varphi) = \frac{E}{r(1-\nu)} y_1 + \frac{\nu}{1-\nu} y_2 + \left[\frac{E}{r(1-\nu)}(z_1 + z_2 + 2z_3) + \frac{\nu}{1-\nu}(z_4 + z_5)\right](q_1 + q_2 + q_3)$$

$$-2\left[\frac{E}{r(1-\nu)} z_2 + \frac{E(2+\nu)}{r(1-\nu^2)} z_3 + \frac{\nu}{1-\nu} z_5\right][q_3 \sin^2\theta + (q_1\cos^2\varphi + q_2 \sin^2\varphi)\cos^2\theta]$$

$$-2\left[\frac{E}{r(1-\nu)} z_2 + \frac{E(1+2\nu)}{r(1-\nu^2)} z_3 + \frac{\nu}{1-\nu} z_5\right](q_1 \sin^2\varphi + q_2\cos^2\varphi),$$

$$\sigma_{\varphi\varphi}(r,\theta,\varphi) = \frac{E}{r(1-\nu)} y_1 + \frac{\nu}{1-\nu} y_2 + \left[\frac{E}{r(1-\nu)}(z_1 + z_2 + 2z_3)\right.$$

$$\left.+\frac{\nu}{1-\nu}(z_4 + z_5)\right](q_1 + q_2 + q_3) - 2\left[\frac{E}{r(1-\nu)} z_2 + \frac{E(1+2\nu)}{r(1-\nu^2)} z_3 + \frac{\nu}{1-\nu} z_5\right][q_3 \sin^2\theta$$

$$+ (q_1 \cos^2\varphi + q_2 \sin^2\varphi)\cos^2\theta] - 2\left[\frac{E}{r(1-\nu)} z_2 + \frac{E(2+\nu)}{r(1-\nu^2)} z_3 + \frac{\nu}{1-\nu} z_5\right](q_1 \sin^2\varphi + q_2 \cos^2\varphi),$$

$$\sigma_{r\theta}(r,\theta,\varphi) = z_6(q_3 - q_1\cos^2\varphi - q_2 \sin^2\varphi)\sin 2\theta,$$

$$\sigma_{r\varphi}(r,\theta,\varphi) = z_6(q_1 - q_2)\sin\theta \sin 2\varphi, \quad \sigma_{\theta\varphi}(r,\theta,\varphi) = \frac{E}{r(1+\nu)} z_3(q_1 - q_2)\cos\theta\sin 2\varphi.$$

Thus, the initial problem (1.1)–(1.5) has reduced to the solution of the boundary-value problems (2.6) and (2.11) in the unknown vector functions $\mathbf{Y}(x)$ and $\mathbf{Z}(x)$.

We replace the local restriction (1.12) by the equivalent integral restriction

$$F_1[\alpha, r_1, \mathbf{Y}, \mathbf{Z}] = 0.5 \int_V \{\eta(\ldots) + |\eta(\ldots)|\}\, dV = \int_0^1 G_1(\alpha, r_1, x, \mathbf{Y}, \mathbf{Z}) = 0, \tag{2.12}$$

where V is the volume of the spherical inclusion; by the parity of the function $\eta(\ldots)$ relative to the coordinate planes X_1OX_2, X_2OX_3, and X_1OX_3, we have the function

$$G_1(\alpha, r_1, x, \mathbf{Y}, \mathbf{Z}) = 4(r_2 - r_1)\, r^2 \int_0^{\pi/2}\int_0^{\pi/2} \{\eta(\ldots) + |\eta(\ldots)|\}\sin\theta\, d\varphi\, d\theta. \tag{2.13}$$

The functional (2.12) has a Frechet derivative [3], because the function $|\eta(\ldots)|$, which is the modulus from the Mises yield condition, can vanish only at a finite number of points, i.e., on a set of zero measure, in the layered spherical inclusion.

Let $\{\alpha(x),\, r_1\}$ be the optimum control from the admissible set (1.8) and (1.9) that minimizes the functional (1.10) and satisfies restriction (2.12). We consider the perturbed control $\{\alpha^*(x),\, r_1+\delta r_1\}$ [3]

$$\alpha^*(x) = \begin{cases} g(x), & x \in D, \quad g(x) \in U, \\ \alpha(x), & x \notin D, \quad \mathrm{mes}(D) < \varepsilon, \end{cases} \qquad r_1 + \delta r_1 \in [a,\, b], \quad |\delta r_1| < \varepsilon, \tag{2.14}$$

where $D \subset [0,\, 1]$ is a set of small measure and $\varepsilon > 0$ is a small quantity.

Using standard techniques [3], one can obtain the principal parts of the increments of the functionals (1.10) and (2.12) [for brevity the arguments of the functions associated with the unperturbed control $\{\alpha(x),\, r_1\}$ are omitted]:

$$\begin{aligned} \delta F[\ldots] &= \int_D \{G(\alpha^*,\ldots) - G(\alpha,\ldots)\}\, dx + S\delta r_1, \\ \delta F_1[\ldots] &= \int_D \{M(\alpha^*,\ldots) - M(\alpha,\ldots)\}\, dx + S_1\delta r_1. \end{aligned} \tag{2.15}$$

Here

$$\begin{aligned} M(\alpha, r_1, x, \mathbf{Y}, \mathbf{Z}, \mathbf{\Phi}, \mathbf{\Psi}) &= G_1(\alpha, r_1, x, \mathbf{Y}, \mathbf{Z}) + \mathbf{\Phi}^t(x)\, A(\alpha, r_1, x)\mathbf{Y}(x) + \mathbf{\Psi}^t(x)\, B(\alpha, r_1, x)\, \mathbf{Z}(x), \\ S = \int_0^1 \frac{\partial}{\partial r_1} G(\alpha, r_1, x)\, dx, &\quad S_1 = \int_0^1 \frac{\partial}{\partial r_1} M(\alpha, r_1, x, \mathbf{Y}, \mathbf{Z}, \mathbf{\Phi}, \mathbf{\Psi})\, dx. \end{aligned} \tag{2.16}$$

The vectors of conjugate variables $\mathbf{\Phi}(x) = (\vartheta_1, \vartheta_2)^t$ and $\mathbf{\Psi}(x) = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6)^t$ satisfy the boundary-value problems

$$\begin{aligned} \mathbf{\Phi}'(x) &= -A^t(\alpha, r_1, x)\mathbf{\Phi}(x) - \left[\frac{\partial}{\partial \mathbf{Y}} G_1(\alpha, r_1, x, \mathbf{Y}, \mathbf{Z})\right]^t, \\ \vartheta_1(0) &= 0, \quad \vartheta_2(1) - \frac{r_2(1+\nu_m)}{2E_m}\vartheta_1(1) = 0; \end{aligned} \tag{2.17}$$

$$\begin{aligned} \mathbf{\Psi}'(x) &= -B^t(\alpha, r_1, x)\, \mathbf{\Psi}(x) - \left[\frac{\partial}{\partial \mathbf{Z}} G_1(\alpha, r_1, x, \mathbf{Y}, \mathbf{Z})\right]^t, \\ \psi_1(0) &= \psi_2(0) = \psi_3(0) = 0, \quad \mathbf{\Psi}_l(1) + C^t(E_m, \nu_m)\, \mathbf{\Psi}_f(1) = 0. \end{aligned} \tag{2.18}$$

We write the expanded functional

$$J[\alpha, r_1] = F[\alpha, r_1] + \lambda_1 F_1[\alpha, r_1, \mathbf{Y}, \mathbf{Z}] + \lambda_2\{a - r_1 + \xi_1^2\} + \lambda_3\{r_1 - b + \xi_2^2\}, \tag{2.19}$$

where $\lambda_1, \lambda_2, \lambda_3$ and ξ_1, ξ_2 are the Lagrange multipliers and the penalty variables, respectively. Using relations (2.15) and (2.16), one can write the variation of functional (2.19) as

$$\delta J[\ldots] = \int_D \{H(\alpha,\ldots) - H(\alpha^*,\ldots)\}dx + \{S + \lambda_1 S_1 - \lambda_2 + \lambda_3\}\delta r_1 + 2\lambda_2\xi_1\, \delta\xi_1 + 2\lambda_3\xi_2\, \delta\xi_2; \tag{2.20}$$

$$H(\alpha, r_1, x, \mathbf{Y}, \mathbf{Z}, \mathbf{\Phi}, \mathbf{\Psi}) = -G(\alpha, r_1, x) - \lambda_1 M(\alpha, r_1, x, \mathbf{Y}, \mathbf{Z}, \mathbf{\Phi}, \mathbf{\Psi}). \tag{2.21}$$

Since the control $\{\alpha(x), r_1\}$ is optimum (minimum), the condition $\delta J[\ldots] \geqslant 0$ should be satisfied for any admissible control $\{\alpha^*(x), r_1 + \delta r_1\}$ (2.14). By virtue of the arbitrary character of the variations δr_1 and $\delta\xi_i$, we have from relation (2.20)

$$S + \lambda_1 S_1 - \lambda_2 + \lambda_3 = 0; \tag{2.22}$$

$$\lambda_2(a - r_1) = 0, \quad \lambda_3(r_1 - b) = 0, \quad \lambda_2 \geqslant 0, \quad \lambda_3 \geqslant 0. \tag{2.23}$$

Since the set of small measure D can be closely located almost everywhere on the interval $[0,\, 1]$, the condition of the maximum of the Hamiltonian function $H(\ldots)$ (2.21) in the argument α [3] should be satisfied for almost all $x \in [0, 1]$:

$$H(\alpha, r_1, x, \mathbf{Y}, \mathbf{Z}, \mathbf{\Phi}, \mathbf{\Psi}) = \max_{\alpha^*(x)\in U} H(\alpha^*, r_1, x, \mathbf{Y}, \mathbf{Z}, \mathbf{\Phi}, \mathbf{\Psi}). \tag{2.24}$$

Thus, we have obtained that the optimum control $\{\alpha(x), r_1\}$ and the corresponding optimum trajectories $\mathbf{Y}(x)$ and $\mathbf{Z}(x)$ and vectors of conjugate variables $\mathbf{\Phi}(x)$ and $\mathbf{\Psi}(x)$ should satisfy the boundary-value problems (2.6), (2.11), (2.17), and (2.18), relations (1.8), (1.9), (2.12), and (2.23), and the optimality conditions (2.22) and (2.24).

3. Computational Algorithm. The main idea of the direct method of solving optimum-design problems is to construct a sequence of controls $\{\alpha(x), r_1\}_j$ ($j = 1, 2, \ldots$) that minimizes the desired functional (1.10). To do this, we introduce a uniform grid $\{x_i\}$ by dividing the interval [0, 1] into n intervals D_i modeling sets of small measure. We give the initial control $\{\alpha(x), r_1\}$ from the admissible domain (1.8), (1.9), and (2.12). Clearly, the function $\alpha(x)$ is piecewise-constant with intervals of constancy $D_i = [x_i, x_{i+1})$ on which this function assumes values from the set U (1.8). The subsequent approximation $\{\alpha^*(x), r_1 + \delta r_1\}$ on a certain set D_i is sought in the form (2.14):

$$\alpha^*(x) = \begin{cases} \alpha_j, & x \in D_i, \quad \alpha_j \in U, \\ \alpha(x), & x \notin D_i; \end{cases} \tag{3.1}$$

$$r_1 + \delta r_1 \in [a, b], \qquad |\delta r_1| < \varepsilon \tag{3.2}$$

and is determined from the linearized optimization problem: to find an admissible perturbation $\{\alpha_j, \delta r_1\}$ on a given set such that it ensures a maximum decrease in the functional $F[\ldots]$ (1.10), i.e., a minimum of the variation $\delta F[\ldots]$ (2.15), under conditions (3.1) and (3.2) and the linearized restriction (2.12)

$$F_1[\alpha^*, r_1 + \delta r_1, \mathbf{Y} + \delta\mathbf{Y}, \mathbf{Z} + \delta\mathbf{Z}] \approx F_1[\alpha, r_1, \mathbf{Y}, \mathbf{Z}] + \delta F_1[\alpha, r_1, \mathbf{Y}, \mathbf{Z}] = 0, \tag{3.3}$$

where the expression for $\delta F_1[\ldots]$ is given by formula (2.15). This linearized problem is a variant of the problem considered in Secs. 1 and 2. Hence we directly obtain that the optimum perturbation $\{\alpha_j, \delta r_1\}$ should satisfy the relations

$$\delta r_1 = -\gamma\{S + \lambda_1 S_1 - \lambda_2 + \lambda_3\}, \quad \gamma \geqslant 0; \tag{3.4}$$

$$\lambda_2(a - r_1 - \delta r_1) = 0, \quad \lambda_3(r_1 + \delta r_1 - b) = 0, \quad \lambda_2 \geqslant 0, \quad \lambda_3 \geqslant 0 \tag{3.5}$$

and restrictions (3.2) and (3.3).

In the process of numerical calculation, the Lagrange multipliers γ, λ_2, and λ_3 are found from (3.2) and (3.5). The optimum correction α_j (3.1) is determined as follows. At $S_1 \neq 0$, we have from relation (3.3)

$$\delta r_1 = -\Big\{ \int_{D_i} [M(\alpha_j, \ldots) - M(\alpha, \ldots)]\, dx + F_1[\alpha, r_1, \mathbf{Y}, \mathbf{Z}] \Big\} \Big/ S_1. \tag{3.6}$$

Substituting (3.6) into $\delta F[\ldots]$ (2.15), we find a correction α_j that minimizes the variation $\delta, F[\ldots]$ from the condition

$$\int_{D_i} H(\alpha_j, r_1, x, \mathbf{Y}, \mathbf{Z}, \mathbf{\Phi}, \mathbf{\Psi})\, dx = \max_{\alpha_* \in U} \int_{D_i} H(\alpha_*, r_1, x, \mathbf{Y}, \mathbf{Z}, \mathbf{\Phi}, \mathbf{\Psi})\, dx,$$

where

$$H(\alpha_*, r_1, x, \mathbf{Y}, \mathbf{Z}, \mathbf{\Phi}, \mathbf{\Psi}) = -G(\alpha_*, r_1, x) + \frac{S}{S_1} M(\alpha_*, r_1, x, \mathbf{Y}, \mathbf{Z}, \mathbf{\Phi}, \mathbf{\Psi}).$$

For $S_1 = 0$, the optimum correction $\{\alpha_j, \delta r_1\}$ is determined from relation (3.4) and the minimum condition for the variation $\delta F[\ldots]$ (2.15):

$$\delta r_1 = -\gamma\{S - \lambda_2 + \lambda_3\}, \quad \int_{D_i} G(\alpha_j, r_1, x)\, dx = \min_{\alpha_* \in U} \int_{D_i} G(\alpha_*, r_1, x)\, dx,$$

TABLE 1

Material	ρ	E	ν	σ_s
Spheroplastic	0.65	270	0.27	4.5
Duralumin	2.85	7100	0.33	44
Titanium alloy	4.6	12000	0.32	80
Steel	7.8	21000	0.3	120
Copper	8.93	11200	0.33	20

with allowance for restrictions (3.2), (3.3), and (3.5).

Having constructed the new control $\{\alpha^*(x), r_1 + \delta r_1\}$, we take it as the initial one and construct the next approximation. The process is assumed to end for a given partition grid $\{x_i\}$ if the control $\{\alpha(x), r_1\}$ changes in none of the sets D_i. The solution obtained is a local minimum in the problem considered.

Example. The set W consists of five materials. The dimensionless mechanical and physical characteristics of these materials are listed in Table 1.

The pressure $p = 0.01$ is set on the inner surface of the spherical inclusion. The inner radius r_1 of the inclusion can vary within the interval [0.75, 0.95], and the outer radius r_2 is considered fixed and equal to unity. The inclusion-containing matrix consists of spheroplastic and is loaded at infinity by the uniform axial forces $q_1 = 4$, $q_2 = 0$, and $q_3 = -4$, i.e., the matrix is subjected to simple shear at infinity. The region of the inclusion is partitioned into 50 sections equal in thickness modeling the sets D_i.

The variations of the control in the above computational algorithm have a local character, i.e., the control changes in just one of the elementary intervals (the set D_i) in each iteration. As is known, such a variation can lead to a deadlock: the structure can be nonoptimum and it is impossible to improve it by a local variation. Therefore, we used various thickness distributions of the materials of the inclusion to be optimized. Based on computational results and some mechanical considerations, we chose new initial approximations, etc. As a result, we obtained a four-layer inclusion of inner radius $r_1 = 0.75123$ and weight $F^* = 8.16$ with layers [0.75123, 0.77611] and [0.82088, 0.92537] of titanium alloy, [0.77611, 0.82088] of spheroplastic, and [0.92537, 1] of Duralumin. The lightest homogeneous inclusion that satisfies the restrictions on tensile strength (1.12) and body thickness (1.9) under prescribed loads p, q_1, q_2, and q_3 is an inclusion made of titanium alloy of inner radius $r_1 = 0.80813$ and weight $F_* = 9.099$. The relative gain in weight for an optimum inclusion compared with this homogeneous inclusion was equal to $(1 - F^*/F_*) \cdot 100\% = 10.3\%$.

This work was supported by the Russian Foundation for Fundamental Research (Grant No. 96-01-01527).

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